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Abstract
We discuss Coleman's theorem concerning the energy density of the ground state of the sine-Gordon model proved in Coleman S (1975 *Phys. Rev.* D **11** 2088). According to this theorem the energy density of the ground state of the sine-Gordon model should be unbounded from below for coupling constants $\beta^2 > 8\pi$. The consequence of this theorem would be the non-existence of the quantum ground state of the sine-Gordon model for $\beta^2 > 8\pi$. We show that the energy density of the ground state in the sine-Gordon model is bounded from below even for $\beta^2 > 8\pi$. This result is discussed in relation to Coleman's theorem (Coleman S 1973 *Commun. Math. Phys.* **31** 259), particle mass spectra and soliton–soliton scattering in the sine-Gordon model.

PACS numbers: 11.10.Ef, 11.10.Gh, 11.10.Kk

1. Introduction

As has been shown in [1, 2] the massless Thirring model is unstable under spontaneous breaking of chiral $U(1) \times U(1)$ symmetry. The non-perturbative phase of spontaneously broken chiral symmetry is described by a ground-state wavefunction of BCS-type [1].

The Lagrangian of the massless Thirring model is given by [1–3]

$$\mathcal{L}_{\mathrm{Th}}(x) = \bar{\psi}(x)\mathrm{i}\gamma^\mu \partial_\mu \psi(x) - \tfrac{1}{2}g\bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(x)\gamma_\mu \psi(x) + \sigma(x)\bar{\psi}(x)\psi(x) \tag{1.1}$$

where $\sigma(x)$ is an external source of the scalar density $\bar{\psi}(x)\psi(x)$ of the Thirring fermion fields and g is the coupling constant, which we treat in the attractive case. For $\sigma(x) = -m$ [2],

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where m can be interpreted as a mass of Thirring fermion fields, the Thirring model (1.1) bosonizes to the sine-Gordon model with the Lagrangian [1, 2]

$$\mathcal{L}_{SG}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0}{\beta^2}(\cos\beta\vartheta(x) - 1) \tag{1.2}$$

where α_0 and β are positive parameters [1, 2, 4]. The parameter α_0 has the meaning of a squared mass of the quantum of the sine-Gordon field

$$\begin{aligned}\mathcal{L}_{SG}(x) &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0}{\beta^2}(\cos\beta\vartheta(x) - 1) \\ &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}\alpha_0\vartheta^2(x) + \frac{1}{4!}\alpha_0\beta^2\vartheta^4(x) + \cdots\end{aligned} \tag{1.3}$$

and β is a coupling constant. For the Thirring fermion fields quantized in the chirally broken phase the coupling constants g and β are related by [1]

$$\frac{8\pi}{\beta^2} = 1 - e^{-2\pi/g}. \tag{1.4}$$

The direct consequence of this relation is that $\beta^2 > 8\pi$. As has been discussed in [1], the relation $\beta^2 > 8\pi$ leads to a (1+1)-dimensional world populated mainly by soliton and antisoliton states [1], which are classical solutions of the equations of motion of the sine-Gordon model (1.2)

$$\Box\vartheta(x) + \frac{\alpha_0}{\beta}\sin\beta\vartheta(x) = 0 \tag{1.5}$$

regardless of the value of the coupling constant β. It is well known that there exists an infinite set of dynamical many-soliton solutions of (1.5) which are collective excitations of the sine-Gordon field [5].

As an example, the one-soliton and one-antisoliton solutions $\vartheta_s(x^0, x^1)$ and $\vartheta_{\bar{s}}(x^0, x^1)$

$$\begin{aligned}\vartheta_s(x^0, x^1) &= \frac{4}{\beta}\arctan(\exp(+\sqrt{\alpha_0}\gamma(x^1 - ux^0))) \\ \vartheta_{\bar{s}}(x^0, x^1) &= \frac{4}{\beta}\arctan(\exp(-\sqrt{\alpha_0}\gamma(x^1 - ux^0)))\end{aligned} \tag{1.6}$$

where u is their velocity and $\gamma = 1/\sqrt{1 - u^2}$ is the Lorentz factor, have a finite classical mass, $M_s = M_{\bar{s}} = 8\sqrt{\alpha_0}/\beta^2$, and are not related to the quantum ground state of the sine-Gordon model.

In his pioneering paper [4], Coleman has proved the equivalence between the massive Thirring model and the sine-Gordon model. A lateral result of Coleman's paper [4] was the proof of the theorem asserting that for $\beta^2 > 8\pi$ the energy density of the sine-Gordon model is unbounded from below. Due to this Coleman argued 'The theory has no ground state, and is physically nonsensical' [4]. In this paper we discuss critically this theorem of Coleman and show that the energy of the ground state of the sine-Gordon model is bounded even for $\beta^2 > 8\pi$.

This paper is organized as follows. In section 2 we repeat Coleman's derivation of the theorem asserting the non-existence of the ground state in the sine-Gordon model for $\beta^2 > 8\pi$ and accentuate those places where we do not agree with Coleman. We modify Coleman's derivation and get a bounded energy density for the ground state of the sine-Gordon model for $\beta^2 > 8\pi$. In section 3 we adduce the explicit calculation of the energy density for the ground state of the sine-Gordon model using the path-integral approach. In sections 4–6 we discuss the relation of the constraint on the coupling constants $\beta^2 > 8\pi$ to (i) Coleman's theorem, asserting the non-existence Goldstone bosons in (1 + 1)-dimensional quantum field theories,

to (ii) particle mass spectra of the sine-Gordon model and to (iii) soliton–soliton scattering. In the conclusion we discuss the obtained results. In the appendix we follow [2] and evaluate the generating functional of Green functions in the sine-Gordon model and demonstrate the infrared stability and non-perturbative renormalizability of this model.

2. Coleman's proof of the theorem on the unbounded vacuum energy density for $\beta^2 > 8\pi$

According to the Lagrangian (1.2) the Hamiltonian of the sine-Gordon model should be equal to

$$\mathcal{H}_{\text{SG}}(x) = \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 - \frac{\alpha_0}{\beta^2}(\cos\beta\vartheta(x) - 1) \tag{2.1}$$

where $\Pi(x) = \dot{\vartheta}(x)$ is the conjugate momentum of the ϑ-field. Following Coleman [4] we transcribe the Hamiltonian (2.1) into the form

$$\mathcal{H}_{\text{SG}}(x) = \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 - \frac{\alpha_0}{\beta^2}\cos\beta\vartheta(x) - \gamma_0 \tag{2.2}$$

where γ_0 is an arbitrary constant, which is equal to

$$\gamma_0 = -\frac{\alpha_0}{\beta^2} \tag{2.3}$$

if the minimum of the classical potential energy is normalized to zero [4].

    The aim of this section is two-fold corresponding to two scenarios of the evolution of the sine-Gordon field. In the first scenario the parameter γ_0 is arbitrary and additively renormalizable, as has been assumed by Coleman. We show that in this scenario the ground state of the sine-Gordon model suffers from an *infrared disaster*. In the infrared limit the renormalized energy density of the ground state of the sine-Gordon model is equal to negative infinity at any coupling constant β. This corresponds to the non-existence of the sine-Gordon model. Hence, Coleman's proof, when analysed with respect to the infrared stability of the sine-Gordon model, leads to the suppression of the sine-Gordon model as quantum field theory. In the second scenario the parameter γ_0 is fixed to the value (2.3) that normalizes the potential energy to zero. In this case the parameter γ_0 is not additively renormalizable. This results in the energy density of the ground state of the sine-Gordon model to be (i) positive definite in the infrared limit and (ii) stable for any coupling constant β even if $\beta^2 > 8\pi$.

    First we analyse the stability of the sine-Gordon model following the scenario when γ_0 is an additively renormalizable parameter. Introducing the infrared scale μ, which should be finally taken in the limit $\mu \to 0$, we can redefine the interaction term in the Hamiltonian (2.2) as follows [4]

$$\cos\beta\vartheta(x) = \left(\frac{\mu^2}{\Lambda^2}\right)^{\beta^2/8\pi} :\cos\beta\vartheta(x):_\mu \tag{2.4}$$

where the symbol $:\ldots:_\mu$ means normal ordering at the scale μ and Λ is the ultraviolet cut-off. Expression (2.4) is a trivial consequence of the perturbative derivation of the vacuum expectation value of the operator $\cos\beta\vartheta(x)$

$$\langle 0|\cos\beta\vartheta(x)|0\rangle = \text{e}^{-\frac{1}{2}\beta^2 D^{(+)}(0;\mu)} = \left(\frac{\mu^2}{\Lambda^2}\right)^{\beta^2/8\pi} \tag{2.5}$$

where $D^{(+)}(x; \mu)$ is the two-point Wightman function defined by

$$D^{(+)}(x; \mu) = \langle 0|\vartheta(x)\vartheta(0)|0\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} e^{-ikx} = -\frac{1}{4\pi}\ln[-\mu^2 x^2 + i0 \cdot \varepsilon(x^0)].$$

(2.6)

For $x = (x^0, x^1) = 0$ the two-point Wightman function is regularized by the ultraviolet cut-off Λ, $|k^1| \leqslant \Lambda$, and reads

$$D^{(+)}(0; \mu) = \frac{1}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right).$$

(2.7)

Since the vacuum expectation value of the normal-ordered operator $: \cos\beta\vartheta(x) :$ is unity, $\langle 0| : \cos\beta\vartheta(x) : |0\rangle = 1$, relation (2.5) can be represented in the operator form (2.4). Of course, the same result can be obtained by considering the ϑ-field as a free field and applying Wick's theorem [4, 6].

Assuming multiplicative renormalizability of the sine-Gordon model Coleman (i) introduces the renormalized constant α determined by

$$\alpha = \alpha_0 \left(\frac{\mu^2}{\Lambda^2}\right)^{\beta^2/8\pi}$$

(2.8)

and (ii) changes the scale of the normal ordering $\mu \to M$ according to the recipe [4]

$$: \cos\beta\vartheta(x) :_\mu \to \left(\frac{M^2}{\mu^2}\right)^{\beta^2/8\pi} : \cos\beta\vartheta(x) :_M .$$

(2.9)

As a result the interaction term of the Hamiltonian of the sine-Gordon model acquires the form

$$\mathcal{H}_{SG}^{int}(x) = -\frac{\alpha}{\beta^2} \left(\frac{M^2}{\mu^2}\right)^{\beta^2/8\pi} : \cos\beta\vartheta(x) :_M$$

(2.10)

where the parameter α is related to the *bare* parameter α_0 by equation (2.8). This completes the redefinition of the interaction part of the Hamiltonian (2.2).

Now according to Coleman we rewrite the free part of the Hamiltonian (2.2) as follows

$$\mathcal{H}_{SG}^{(0)}(x) = \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 = : \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 :_\mu + \mathcal{E}_0(\mu)$$

(2.11)

where $\mathcal{E}_0(\mu)$ is equal to [4]

$$\mathcal{E}_0(\mu) = \int_{-\infty}^{\infty} \frac{dk^1}{8\pi} \frac{2(k^1)^2 + \mu^2}{\sqrt{(k^1)^2 + \mu^2}}.$$

(2.12)

The regularized version of $\mathcal{E}_0(\mu)$ reads

$$\mathcal{E}_0(\Lambda, \mu) = \int_{-\Lambda}^{\Lambda} \frac{dk^1}{8\pi} \frac{2(k^1)^2 + \mu^2}{\sqrt{(k^1)^2 + \mu^2}} = \frac{\Lambda^2}{4\pi}\sqrt{1 + \frac{\mu^2}{\Lambda^2}} = \frac{\Lambda^2}{4\pi} + \frac{\mu^2}{8\pi} + O\left(\frac{\mu^4}{\Lambda^2}\right).$$

(2.13)

The appearance of $\mathcal{E}_0(\Lambda, \mu)$ can easily be justified using the expansions of the field $\vartheta(x)$ and the conjugate momentum $\Pi(x)$ into plane waves [4, 7]

$$\vartheta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} (a(k^1) e^{-ikx} + a^\dagger(k^1) e^{ikx})$$

$$\Pi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2i} (a(k^1) e^{-ikx} - a^\dagger(k^1) e^{ikx})$$

(2.14)

where $k^0 = \sqrt{(k^1) + \mu^2}$, $a(k^1)$ and $a^\dagger(k^1)$ are annihilation and creation operators obeying the standard commutation relation

$$[a(k^1), a^\dagger(q^1)] = (2\pi)2k^0\delta(k^1 - q^1). \tag{2.15}$$

Assuming additive renormalizability of the parameter γ_0 Coleman defines the renormalized parameter γ

$$\gamma = \gamma_0 + \mathcal{E}_0(\mu) \tag{2.16}$$

and redefines the free part of the Hamiltonian (2.2) as follows:

$$\mathcal{H}_{SG}^{(0)}(x) - \gamma_0 = {:}\mathcal{H}_{SG}^{(0)}(x){:}_\mu - \gamma. \tag{2.17}$$

After this set of transformations Coleman asserts that 'Assembling all this, we find the cut-off independent form of the Hamiltonian density':

$$\mathcal{H}_{SG}(x) = {:}\frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 - \frac{\alpha}{\beta^2}\cos\beta\vartheta(x){:}_\mu - \gamma \tag{2.18}$$

where all parameters α, β and γ are finite.

Changing then the scale of the normal ordering $\mu \to M$ [4]

$${:}\mathcal{H}_{SG}^{(0)}(x){:}_\mu \to {:}\mathcal{H}_{SG}^{(0)}(x){:}_M + \mathcal{E}_0(M) - \mathcal{E}_0(\mu) = {:}\mathcal{H}_{SG}^{(0)}(x){:}_M + \frac{1}{8\pi}(M^2 - \mu^2) \tag{2.19}$$

Coleman arrives at the Hamiltonian

$$\mathcal{H}_{SG}(x) = {:}\frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 {:}_M - \frac{\alpha}{\beta^2}\left(\frac{M^2}{\mu^2}\right)^{\beta^2/8\pi} {:}\cos\beta\vartheta(x){:}_M$$
$$+ \frac{1}{8\pi}(M^2 - \mu^2) - \gamma. \tag{2.20}$$

The renormalized energy density of the ground state is equal to

$$\mathcal{E}_{vac}(M) = -\frac{\alpha}{\beta^2}\left(\frac{M^2}{\mu^2}\right)^{\beta^2/8\pi} + \frac{1}{8\pi}(M^2 - \mu^2) - \gamma. \tag{2.21}$$

This is equation (3.7) of [4]. The renormalized energy density $\mathcal{E}_{vac}(M)$ (2.21) depends explicitly on the infrared cut-off μ, which should be taken in the limit $\mu \to 0$, whereas all parameters α, β and γ are kept finite. Before one analyses the behaviour of the renormalized energy density in the limit $M \to \infty$, one has to take the limit $\mu \to 0$. Taking the limit $\mu \to 0$ one gets the renormalized energy density (2.21) equal to negative infinity for any finite scale M and any coupling constant $\beta \neq 0$. This makes the sine-Gordon model an extremely ill-defined quantum field theory and *nonsensical* for any coupling constant $\beta \neq 0$. This contradicts the infrared stability of the sine-Gordon model (see, for example, the appendix this paper) and gives no constraints on the value of the coupling constant like $\beta^2 < 8\pi$.

Such an *infrared disaster* is a consequence of two of Coleman's assumptions, the finiteness of the parameter α and the additive renormalizability of the parameter γ. The finiteness of the parameter α in (2.8) is questionable, since this entails the infinity of the parameter α_0 in the infrared limit $\mu \to 0$. The former is not really true. Indeed, as we have shown (see the appendix in this paper) the sine-Gordon model is non-singular in the infrared limit (A.12) and the correlation functions of the sine-Gordon model are finite in this limit. Moreover, if α_0 would be infinite in the infrared limit the soliton solutions of the sine-Gordon model such as (1.6) would not exist. Hence, there is no physical reason for the parameter α_0 to be infinite at $\mu \to 0$.

Thus, Coleman's scenario of the evolution of the sine-Gordon field with an arbitrary and additively renormalizable parameter γ_0 leads to the *infrared disaster* of the sine-Gordon model and makes no constraints on the value of the coupling constant β.

Now let us analyse another scenario of the evolution of the sine-Gordon field with a potential energy normalized to zero. In this case the parameter γ_0 is fixed to the value (2.3) and after the renormalization of the parameter α_0 (2.8) one should get the renormalized γ, i.e.

$$\gamma = -\frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{\mu^2}\right)^{\beta^2/8\pi}. \tag{2.22}$$

This yields the Hamiltonian (2.18) depending explicitly on the ultraviolet cut-off Λ

$$\mathcal{H}_{\text{SG}}(x) = :\frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 :_\mu - \frac{\alpha}{\beta^2} : \cos\beta\vartheta(x) :_\mu + \frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{\mu^2}\right)^{\beta^2/8\pi} + \mathcal{E}_0(\Lambda,\mu). \tag{2.23}$$

Since unlike Coleman through the parameter γ_0 the Hamiltonian depends on the ultraviolet cut-off Λ, one does not need to remove the ultraviolet divergence of $\mathcal{E}_0(\Lambda,\mu)$ appearing due to the normal ordering of the kinetic term.

Following then Coleman and changing the scale of the normal ordering $\mu \to M$ we arrive at the Hamiltonian

$$\mathcal{H}_{\text{SG}}(x) = :\frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 :_M - \frac{\alpha}{\beta^2}\left(\frac{M^2}{\mu^2}\right)^{\beta^2/8\pi} : \cos\beta\vartheta(x) :_M$$
$$+ \frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{\mu^2}\right)^{\beta^2/8\pi} + \mathcal{E}_0(\Lambda, M). \tag{2.24}$$

The vacuum energy density defined by the Hamiltonian (2.24) is equal to

$$\mathcal{E}_{\text{vac}}(M) = \frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{\mu^2}\right)^{\beta^2/8\pi} - \frac{\alpha}{\beta^2}\left(\frac{M^2}{\mu^2}\right)^{\beta^2/8\pi} + \frac{\Lambda^2}{4\pi}\sqrt{1 + \frac{M^2}{\Lambda^2}}. \tag{2.25}$$

Due to the first term in the rhs of (2.25), which is absent in Coleman's expression given by equation (3.7) of [4], the energy density of the ground state of the sine-Gordon model is positive definite in the infrared limit. At $M = \Lambda$ the energy density does not depend on the infrared cut-off and is proportional to $\mathcal{E}_{\text{vac}}(\Lambda) \sim \Lambda^2$. The problem of this quadratic ultraviolet divergence can easily be solved taking the full Hamiltonian (2.1) in the normal-ordered form.

Thus, we argue that the sine-Gordon model with the potential energy normalized to zero is stable in the infrared limit and the energy of the ground state can never be negative for arbitrary values of the coupling constant β even for $\beta^2 > 8\pi$.

3. Vacuum energy density in the sine-Gordon model: non-perturbative calculation

Using the Lagrangian $\mathcal{L}_{\text{SG}}(x)$ given by (1.2) one can obtain the Hamilton functional $H(x^0)$ of the sine-Gordon model

$$H(x^0) = \int_{-\infty}^{\infty} dx^1 \left\{\frac{1}{2} : \left[\Pi^2(x) + \left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2\right] : -\frac{\alpha_0}{\beta^2} : [\cos\beta\vartheta(x) - 1] : \right\}. \tag{3.1}$$

The first two terms describe the contribution of the kinetic energy which should be always taken in the normal-ordered form[2]. In quantum field theory the potential energy, given by the last two terms in (3.1), should be normal ordered as well as the kinetic one. However, below we consider two possibilities (i) the potential energy is normal-ordered and (ii) the potential energy is not normal-ordered. We will show that in the case of the potential energy, taken in the normal-unordered form, the energy of the ground state of the sine-Gordon model tends to positive infinity.

The energy of the ground state E_{vac} is equal to the vacuum expectation value of the Hamilton functional $H(x^0)$

$$E_{\text{vac}} = \langle 0|H(x^0)|0\rangle = -\frac{\alpha_0}{\beta^2} \int_{-\infty}^{\infty} dx^1 [\langle 0| \cos \beta \vartheta(x)|0\rangle - 1]. \tag{3.2}$$

Since the integrand in (3.2) does not depend on x, instead of the energy of the ground state E_{vac} it is convenient to treat the vacuum energy density $\mathcal{E}_{\text{vac}}$ defined by

$$\mathcal{E}_{\text{vac}} = \lim_{L \to \infty} \frac{E_{\text{vac}}}{L} = -\frac{\alpha_0}{\beta^2} [\langle 0| \cos \beta \vartheta(0)|0\rangle - 1] \tag{3.3}$$

where L is the spatial volume.

 (i) If the potential energy is taken in the normal-ordered form, the energy density $\mathcal{E}_{\text{vac}}$ is equal to zero, $\mathcal{E}_{\text{vac}} = 0$, due to $\langle 0| : \cos \beta \vartheta(0) : |0\rangle = 1$ by definition of the normal ordering.
(ii) In the case of the normal-unordered form of the potential energy the vacuum expectation value $\langle 0|[\cos \beta \vartheta(0) - 1]|0\rangle$ is non-zero and can be calculated explicitly. In terms of the partition function $Z_{\text{SG}}[0]$ defined by (A.15) the vacuum expectation value $\langle 0|[\cos \beta \vartheta(0) - 1]|0\rangle$ reads

$$\frac{\alpha_0}{\beta^2} [\langle 0| \cos \beta \vartheta(0)|0\rangle - 1] = \lim_{T,L \to \infty} \frac{1}{TL} \frac{\alpha_0}{i} \frac{\partial \ln Z_{\text{SG}}[0]}{\partial \alpha_0} \tag{3.4}$$

where TL defines a $(1+1)$-dimensional volume, $\int d^2x = \int dx^0 \, dx^1 = TL$, at $T, L \to \infty$.

By the renormalization $\alpha_0 \to Z_1\alpha$, where Z_1 is a renormalization constant (A.13), we obtain

$$\frac{\alpha_0}{\beta^2} [\langle 0| \cos \beta \vartheta(0)|0\rangle - 1] = \lim_{T,L \to \infty} \frac{1}{TL} \frac{\alpha}{i} \frac{\partial \ln Z_{\text{SG}}[0]}{\partial \alpha}. \tag{3.5}$$

Substituting (3.5) in (3.3) we determine the vacuum energy density $\mathcal{E}_{\text{vac}}$ in terms of the partition function $Z_{\text{SG}}[0]$ as follows:

$$\mathcal{E}_{\text{vac}} = -\lim_{T,L \to \infty} \frac{1}{TL} \frac{\alpha}{i} \frac{\partial \ln Z_{\text{SG}}[0]}{\partial \alpha}. \tag{3.6}$$

Due to (A.15) the vacuum energy density $\mathcal{E}_{\text{vac}}$ is equal to

$$\mathcal{E}_{\text{vac}} = \frac{\alpha}{\beta^2} \left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi} + \frac{2}{i} \lim_{T,L \to \infty} \frac{1}{TL} \sum_{p=1}^{\infty} \frac{(-1)^p}{p!(p-1)!} \left(\frac{\alpha}{2\beta^2}\right)^{2p} \prod_{j=1}^{p} \iint d^2x_j \, d^2y_j$$

$$\times \exp\left\{\frac{\beta^2}{4\pi} \sum_{j<k}^{p} (\ln[-M^2(x_j - x_k)^2 + i0] + \ln[-M^2(y_j - y_k)^2 + i0])\right.$$

---

[2] We carry out the normal ordering at the infrared scale μ, which is taken finally in the limit $\mu \to 0$ (see the appendix).

$$
-\frac{\beta^2}{4\pi}\sum_{j=1}^{p}\sum_{k=1}^{p}\ln[-M^2(x_j-y_k)^2+\mathrm{i}0]\Biggr\}\Biggl[\sum_{q=0}^{\infty}\frac{(-1)^q}{(q!)^2}\left(\frac{\alpha}{2\beta^2}\right)^{2q}
$$

$$
\times\prod_{j=1}^{q}\iint \mathrm{d}^2x_j\,\mathrm{d}^2y_j\exp\Biggl\{\frac{\beta^2}{4\pi}\sum_{j<k}^{q}(\ln[-M^2(x_j-x_k)^2+\mathrm{i}0]
$$

$$
+\ln[-M^2(y_j-y_k)^2+\mathrm{i}0])\;-\frac{\beta^2}{4\pi}\sum_{j=1}^{q}\sum_{k=1}^{q}\ln[-M^2(x_j-y_k)^2+\mathrm{i}0]\Biggr\}\Biggr]^{-1}
$$

(3.7)

where the second term is a ratio of two infinite series and M is a finite scale. One can show that this term vanishes in the limit $T, L \to \infty$. For this aim we rewrite the vacuum energy density (3.7) in the form

$$
\mathcal{E}_{\text{vac}}=\frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi}+\frac{2}{\mathrm{i}}\lim_{T,L\to\infty}\left(\frac{\alpha}{2\beta^2}\right)^2\Biggl[-\int\frac{\mathrm{d}^2x}{(-M^2x^2+\mathrm{i}0)^{\beta^2/4\pi}}+\frac{1}{2}\left(\frac{\alpha}{2\beta^2}\right)^2
$$

$$
\times\int\frac{\mathrm{d}^2x\,\mathrm{d}^2y\,\mathrm{d}^2z[(-M^2(x-y)^2+\mathrm{i}0)(-M^2z^2+\mathrm{i}0)]^{\beta^2/4\pi}}{[(-M^2x^2+\mathrm{i}0)(-M^2y^2+\mathrm{i}0)(-M^2(x-y)^2+\mathrm{i}0)(-M^2(y-z)^2+\mathrm{i}0)]^{\beta^2/4\pi}}+\cdots\Biggr]
$$

$$
:\Biggl[1-TL\frac{1}{2}\left(\frac{\alpha}{2\beta^2}\right)^2\int\frac{\mathrm{d}^2x}{(-M^2x^2+\mathrm{i}0)^{\beta^2/4\pi}}+TL\frac{1}{4}\left(\frac{\alpha}{2\beta^2}\right)^4
$$

$$
\times\int\frac{\mathrm{d}^2x\,\mathrm{d}^2y\,\mathrm{d}^2z[(-M^2(x-y)^2+\mathrm{i}0)(-M^2z^2+\mathrm{i}0)]^{\beta^2/4\pi}}{[(-M^2x^2+\mathrm{i}0)(-M^2y^2+\mathrm{i}0)(-M^2(x-y)^2+\mathrm{i}0)(-M^2(y-z)^2+\mathrm{i}0)]^{\beta^2/4\pi}}+\cdots\Biggr].
$$

(3.8)

It is seen that in the limit $T, L \to \infty$ the ratio of the series is of order $O(1/TL)$. This allows us to rewrite (3.8) as follows:

$$
\mathcal{E}_{\text{vac}}=\frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi}+\frac{2}{\mathrm{i}}\lim_{T,L\to\infty}O\left(\frac{1}{TL}\right).
$$

(3.9)

Hence, in the limit $T, L \to \infty$ the vacuum energy density $\mathcal{E}_{\text{vac}}$ is defined only by the first term in (3.8). This gives

$$
\mathcal{E}_{\text{vac}}=\frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi}.
$$

(3.10)

Since the renormalized coupling constant α is finite as well as the coupling constant β, in the limit $\Lambda \to \infty$ the vacuum energy density $\mathcal{E}_{\text{vac}}$ tends to positive infinity as it is usual for renormalizable quantum field theories with Hamilton functionals taken in the normal-unordered form.

We would like to remind that Coleman's expression for the energy density of the ground state of the sine-Gordon model is linear in the coupling constant α_0. Therefore, formally, for the verification of Coleman's result we can consider only the lowest order in perturbation theory with respect to the coupling constant α_0. Taking the potential energy in the normal-unordered form and keeping only the lowest order in the α_0 expansion the vacuum expectation value $\langle 0|\cos\beta\vartheta(0)|0\rangle$ amounts to

$$
\langle 0|\cos\beta\vartheta(0)|0\rangle=\lim_{\mu\to0}\left(\frac{\mu^2}{\Lambda^2}\right)^{\beta^2/8\pi}=0.
$$

(3.11)

This gives the vacuum energy density (3.3) equal to $\mathcal{E}_{\text{vac}} = \alpha_0/\beta^2$, which reduces to (3.10) after renormalization $\alpha_0 = \alpha Z_1 = \alpha(\Lambda^2/M^2)^{\beta^2/8\pi}$ with the renormalization constant $Z_1 = (\Lambda^2/M^2)^{\beta^2/8\pi}$ defined by (A.13).

The vacuum energy density (3.10) tends to infinity at $\Lambda \to \infty$. Such an infinity can be removed by normal ordering. Hence, according to standard conclusions of quantum field theory the energy of the ground state of the sine-Gordon model is equal to zero, if the Hamilton functional is taken in the normal-ordered form.

Within the path-integral approach, where the vacuum energy density of the sine-Gordon model is defined by the generating functional of Green functions $Z_{\text{SG}}[J]$ for the external source zero, $J = 0$, the energy density of the ground state of the sine-Gordon model can be set zero normalizing $Z_{\text{SG}}[J]$ to unity at $J = 0$, i.e. $Z_{\text{SG}}[0] = 1$.

In the following sections we discuss our result for the ground state of the sine-Gordon model to be bounded from below for $\beta^2 > 8\pi$ in relation to (i) Coleman's theorem [10], asserting the absence of Goldstone bosons and spontaneously broken continuous symmetry in quantum field theories in (1 + 1)-dimensional spacetime with Wightman's observables defined on the test functions from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ [11], (ii) particle mass spectra and (iii) soliton–soliton scattering in the sine-Gordon model.

4. Relation to Coleman's theorem 'There are no Goldstone bosons in two dimensions'

The constraint $\beta^2 > 8\pi$ on the coupling constants β appears as a result of the bosonization of the massless Thirring model with fermion fields quantized in the chirally broken phase [1] and the normalization of the Lagrangian of the free massless (pseudo)scalar field $\vartheta(x)$ to the standard form $\mathcal{L}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x)$. Coupling constants $\beta^2 > 8\pi$ define the nonlinear response of the free massless (pseudo)scalar field $\vartheta(x)$ on external sources of Thirring fermion fields. The wavefunction of the ground state of the free massless (pseudo)scalar field has been obtained through the bosonization of the BCS-type wavefunction of the ground state of the massless Thirring model in the chirally broken phase [8]. This wavefunction is not invariant under chiral transformations, related to the constant shifts of the free massless (pseudo)scalar field $\vartheta(x) \to \vartheta(x) + \alpha$, and caused fully by the collective zero mode of this field [7–9]. The collective zero mode of the free massless (pseudo)scalar field $\vartheta(x)$, describing the motion of the 'centre of mass' of the system, is responsible for the infrared divergences of the two-point Wightman functions [7], which lead to the vanishing of the generating functional of Green functions $Z[J]$

$$Z[J] = \int \mathcal{D}\vartheta \exp\left\{ i \int d^2x \left[\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \vartheta(x)J(x) \right] \right\}$$

of the field $\vartheta(x)$, where $J(x)$ is the external source of this field.

The non-vanishing value of $Z[J]$ can be obtained by the removal of the collective zero mode from the spectrum of observable modes. This can be carried out by the constraint on the external source $\int d^2x\, J(x) = \tilde{J}(0) = 0$ [2, 7, 8] (see also (A.3) of the appendix)[3].

As has been pointed out by Wightman [11] the quantum field theory of a free massless (pseudo)scalar field in (1 + 1)-dimensional spacetime does not exist from a mathematical point of view, if Wightman's observables are defined on the test functions $h(x)$ from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. In this case Wightman's positive definiteness condition is violated due to infrared divergences of the two-point Wightman functions [11]. Nevertheless, Wightman

---

[3] Recall that the removal of the collective zero mode from the spectrum of observable modes has been discussed by Hasenfratz [12] in connection with a correct formulation of Feynman rules in one- and two-dimensional nonlinear σ-models with $O(N)$ symmetry.

has argued that the problem of the violation of Wightman's positive definiteness condition can be avoided defining Wightman's observables on the test functions from the Schwartz class $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0\}$, where $\tilde{h}(k)$ is the Fourier transform of the test function $h(x)$. As has been shown in [13] the quantum field theory of the free massless (pseudo)scalar field with Wightman's observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ is equivalent to the quantum field theory determined by the generating functional of Green functions $Z[J]$ with external sources obeying the constraint $\int d^2x \, J(x) = \tilde{J}(0) = 0$. Since the collective zero mode is not induced, such a quantum field theory does not suffer from infrared divergences of the two-point Wightman functions [7].

In [10], Coleman has reformulated Wightman's ban on the construction of the quantum field theory of a free massless (pseudo)scalar field in $(1 + 1)$-dimensional spacetime with Wightman's observables defined on the test functions $h(x) \in \mathcal{S}(\mathbb{R}^2)$ as non-existence of Goldstone bosons, massless (pseudo)scalar fields, and spontaneously broken continuous symmetry in $(1 + 1)$-dimensional quantum field theories. The removal of the collective zero mode from the system allows us to formulate in $(1 + 1)$-dimensional spacetime a consistent quantum field theory of a free massless (pseudo)scalar field without infrared divergences. This quantum field theory is equivalent to Wightman's version of the quantum field theory of a free massless (pseudo)scalar field with Wightman's observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$. Since Coleman's theorem concerns only $(1 + 1)$-dimensional quantum field theories with Wightman's observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$ and tells nothing about the absence of Goldstone bosons and spontaneous breaking of continuous symmetry in quantum field theories with Wightman's observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ [8, 13], the coupling constants, obeying the constraint $\beta^2 > 8\pi$, do not contradict Coleman's theorem [10]. This is because such coupling constants are related to the quantum field theory with Wightman's observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ [8, 13].

The sine-Gordon model has been obtained through the bosonization of the massive Thirring model with fermion fields quantized relative to the non-perturbative BCS-type superconducting vacuum [1]. The constraint $\beta^2 > 8\pi$ on the coupling constant β has appeared naturally due to the normalization of the kinetic term of the Lagrangian of the sine-Gordon field to $\frac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x)$. Therefore, the sine-Gordon field $\vartheta(x)$ has inherited all properties of the free massless (pseudo)scalar field $\vartheta(x)$, bosonizing the massless Thirring model in the chirally broken phase, which have been extended by the inclusion of the sine-Gordon interaction. This means that in our approach the sine-Gordon model is a quantum field theory of a self-coupled (pseudo)scalar field $\vartheta(x)$ with Wightman's observables defined on the test functions from the Schwartz class $\mathcal{S}_0(\mathbb{R}^2)$ (see the appendix).

5. Particle mass spectra

According to Korepin *et al* [15] the sine-Gordon model describes three sorts of particle states with masses: (i) $M_q = \sqrt{\alpha_0}$, (ii) $M_s = M_{\bar{s}} = 8\sqrt{\alpha_0}/\beta^2$ and (iii) $M_{\text{br}}^{(n)} = 2M_s \sin \nu_n$, where $\nu_n = n\beta^2/16$ with $n = 1, 2, \ldots, 8\pi/\beta^2$.

The particles with mass $M_q = \sqrt{\alpha_0}$ are the quanta of the sine-Gordon field in the perturbative regime $\beta^2 \ll 4\pi$, when the potential $V[\vartheta(x)] = (\alpha_0/\beta^2)(1 - \cos \beta \vartheta(x))$ can be expanded in powers of β^2,

$$V[\vartheta(x)] = \frac{1}{2}\alpha_0 \vartheta^2(x) - \frac{1}{24}\alpha_0 \beta^2 \vartheta^4(x) + \cdots. \tag{5.1}$$

These quanta are described by the operators of annihilation and creation in the expansion of the ϑ-field into plane waves such as (2.14).

The particles with mass $M_s = M_{\bar{s}} = 8\sqrt{\alpha_0}/\beta^2$ are single solitons and antisolitons, whose masses do not contain quantum corrections [19]. The single solitons and antisolitons are described by (1.6). The soliton–soliton and the soliton–antisoliton states read [16–18]

$$
\begin{aligned}
\vartheta_{ss}(x^0, x^1) &= \frac{4}{\beta} \tan^{-1} \left(u \frac{\sinh(\sqrt{\alpha_0}\gamma x^1)}{\cosh(\sqrt{\alpha_0}\gamma u x^0)} \right) \\
\vartheta_{s\bar{s}}(x^0, x^1) &= \frac{4}{\beta} \tan^{-1} \left(\frac{1}{u} \frac{\sinh(\sqrt{\alpha_0}\gamma u x^0)}{\cosh(\sqrt{\alpha_0}\gamma x^1)} \right)
\end{aligned}
\tag{5.2}
$$

where $\gamma = 1/\sqrt{1 - u^2}$ is the Lorentz factor.

The total energies of these soliton–soliton and soliton–antisoliton states are equal to $E = 2M_s\gamma$ [16, 18].

The particles with mass $M_{br}^{(n)} = 2M_s \sin \nu_n$ are the *breather* solutions. Breathers describe soliton–antisoliton bound states [19]. In the rest frame the classical solution corresponding to the nth quantum state reads [17, 19]

$$
\vartheta_{br}^{(n)}(x^0, x^1) = \frac{4}{\beta} \tan^{-1} \left(\tan \nu_n \frac{\sin(\sqrt{\alpha_0}x^0 \cos \nu_n)}{\cosh(\sqrt{\alpha_0}x^1 \sin \nu_n)} \right).
\tag{5.3}
$$

As has been shown by Dashen *et al* [19] small quantum fluctuations around a one-soliton solution lead to a change of the soliton (antisoliton) mass as follows

$$
M_s = M_{\bar{s}} = \frac{8\sqrt{\alpha_0}}{\beta^2} - \frac{\sqrt{\alpha_0}}{\pi} = \frac{8\sqrt{\alpha_0}}{\tilde{\beta}^2}
\tag{5.4}
$$

where we have denoted

$$
\tilde{\beta}^2 = \frac{\beta^2}{1 - \beta^2/8\pi}.
\tag{5.5}
$$

The masses of breathers are then changed as $M_{br}^{(n)} = 2M_s \sin \tilde{\nu}_n$, where $\tilde{\nu}_n = n\tilde{\beta}^2/16$ with $n = 1, 2, \ldots, 8\pi/\tilde{\beta}^2$ [19] and M_s given by (5.4).

This contribution of quantum fluctuations to the soliton (antisoliton) mass has been obtained in [19] for $\beta^2 < 8\pi$. At $\beta^2 = 8\pi$ formula (5.5) predicts a singularity.

However, according to Zamolodchikov and Zamolodchikov [20] 'The singularity of the sine-Gordon theory at $\beta^2 = 8\pi \ldots$ scarcely means the failure of the theory with $\beta^2 \geqslant 8\pi$, but rather indicates a lack of superrenormalizability property and suggests that another renormalization prescription is necessary at $\beta^2 \geqslant 8\pi$'.

6. Quantum fluctuations around classical solutions, renormalization and soliton–soliton scattering

The non-perturbative renormalization of the sine-Gordon model has been carried out in [2] (see also the appendix in this paper). We apply this renormalization procedure to the calculation of the contribution of quantum fluctuations around a soliton (antisoliton) solution. The result can be treated as a continuation of the theory to the region of coupling constants with $\beta^2 > 8\pi$. We start with the partition function

$$
\begin{aligned}
Z_{SG} &= \int \mathcal{D}\vartheta \exp \left\{ i \int d^2x \left[\frac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + \frac{\alpha_0}{\beta^2}(\cos \beta\vartheta(x) - 1) \right] \right\} \\
&= \int \mathcal{D}\vartheta \exp \left\{ i \int d^2x \mathcal{L}[\vartheta(x)] \right\}.
\end{aligned}
\tag{6.1}
$$

Following Dashen *et al* [19] we treat the fluctuations of the sine-Gordon field $\vartheta(x)$ around the classical solution $\vartheta(x) = \vartheta_{cl}(x) + \varphi(x)$, where $\vartheta_{cl}(x)$ is any classical solution, satisfying the equations of motion (1.5), and $\varphi(x)$ is the fluctuating field.

Substituting $\vartheta(x) = \vartheta_{cl}(x) + \varphi(x)$ into the exponent of the integrand of (6.1) and using the equations of motion (1.5) for the classical solution $\vartheta_{cl}(x)$ we get

$$Z_{SG} = \exp\left\{i \int d^2x \mathcal{L}[\vartheta_{cl}(x)]\right\} \int \mathcal{D}\varphi \exp\left\{i \int d^2x \left[\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) + \frac{\alpha_0}{\beta}\sin\beta\vartheta_{cl}(x)\varphi(x)\right.\right.$$
$$\left.\left. + \frac{\alpha_0}{\beta^2}(\cos(\beta\vartheta_{cl}(x) + \beta\varphi(x)) - \cos\beta\vartheta_{cl}(x))\right]\right\}. \tag{6.2}$$

In the Gaussian approximation [19] the integrand reads

$$Z_{SG} = \exp\left\{i \int d^2x \mathcal{L}[\vartheta_{cl}(x)]\right\}$$
$$\times \int \mathcal{D}\varphi \exp\left\{i \int d^2x \left[\tfrac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \tfrac{1}{2}\alpha_0\cos\beta\vartheta_{cl}(x)\varphi^2(x)\right]\right\}. \tag{6.3}$$

The exponent of the integral over $\varphi(x)$ coincides with that in equation (3.4) of [19]. Integrating over $\varphi(x)$ we obtain

$$Z_{SG} = \frac{1}{\sqrt{\text{Det}(\Box + \alpha_0\cos\beta\vartheta_{cl})}}\exp\left\{i \int d^2x\, \mathcal{L}[\vartheta_{cl}(x)]\right\}$$
$$= \exp\left\{i \int d^2x\, \mathcal{L}_{eff}[\vartheta_{cl}(x)]\right\} \tag{6.4}$$

where the effective Lagrangian $\mathcal{L}_{eff}[\vartheta_{cl}(x)]$ is defined by

$$\mathcal{L}_{eff}[\vartheta_{cl}(x)] = \mathcal{L}[\vartheta_{cl}(x)] + i\frac{1}{2}\langle x|\ln\left(1 + \frac{\alpha_0}{\Box + \alpha_0}(\cos\beta\vartheta_{cl}(x) - 1)\right)|x\rangle. \tag{6.5}$$

The wavefunctions $|x\rangle$ are normalized by $\langle x|y\rangle = \delta^{(2)}(x - y)$ [1, 21].

The first-order correction to $\mathcal{L}[\vartheta_{cl}(x)]$ is equal to

$$\mathcal{L}^{(1)}[\vartheta_{cl}(x)] = -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2 i}\frac{\alpha_0}{\alpha_0 - k^2 - i0}(\cos\beta\vartheta_{cl}(x) - 1)$$
$$= -\frac{\alpha_0}{8\pi}\int_0^{\Lambda^2}\frac{dk_E^2}{\alpha_0 + k_E^2}(\cos\beta\vartheta_{cl}(x) - 1) = -\frac{\alpha_0}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right)(\cos\beta\vartheta_{cl}(x) - 1) \tag{6.6}$$

where Λ is the ultraviolet cut-off. We carried out the Wick rotation to the Euclidean momentum space $d^2k = id^2k_E$ and $k^2 = -k_E^2$.

For the effective Lagrangian $\mathcal{L}_{eff}[\vartheta_{cl}(x)]$ we obtain

$$\mathcal{L}_{eff}[\vartheta_{cl}(x)] = \frac{1}{2}\partial_\mu\vartheta_{cl}(x)\partial^\mu\vartheta_{cl}(x) + \frac{\alpha_0}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right)\right](\cos\beta\vartheta_{cl}(x) - 1). \tag{6.7}$$

Now we have to renormalize the coupling constant α_0 in order to remove the ultraviolet cut-off Λ. The coupling constant $\alpha(M)$ renormalized at the normalization scale M is defined by $\alpha(M) = Z_1^{-1}(\beta, M, \alpha_0; \Lambda)\alpha_0$ (see the appendix). The renormalization constant $Z_1(\beta, M, \alpha_0; \Lambda)$ is equal to (A.13)

$$Z_1(\beta, M, \alpha_0; \Lambda) = \left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi} = 1 + \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{M^2}\right) + \cdots. \tag{6.8}$$

The renormalization of the effective potential in the effective Lagrangian (6.7) runs as follows. Treating only the constant factor in front of $(\cos \beta \vartheta_{cl}(x) - 1)$ we get

$$\frac{\alpha_0}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right)\right] = \frac{\alpha Z_1}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha Z_1}\right)\right] = \frac{\alpha}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha}\right) + (Z_1 - 1)\right]$$
$$= \frac{\alpha}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha}\right) + \frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{M^2}\right)\right] = \frac{\alpha}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{M^2}{\alpha}\right)\right]$$
(6.9)

where we have dropped the terms of order of $O(\beta^4)$.

Thus, the renormalized effective Lagrangian of the sine-Gordon model reads

$$\mathcal{L}_{\text{eff}}^{(r)}[\vartheta_{cl}(x)] = \frac{1}{2}\partial_\mu \vartheta_{cl}(x)\partial^\mu \vartheta_{cl}(x) + \frac{\alpha}{\beta^2}\left[1 - \frac{\beta^2}{8\pi}\ln\left(\frac{M^2}{\alpha}\right)\right](\cos \beta \vartheta_{cl}(x) - 1)$$
(6.10)

where we have denoted $\alpha = \alpha(M)$.

The non-perturbative correction, caused by quantum fluctuations around a classical solution, to the effective potential of the sine-Gordon model can be written as

$$\mathcal{L}_{\text{eff}}^{(r)}[\vartheta_{cl}(x)] = \frac{1}{2}\partial_\mu \vartheta_{cl}(x)\partial^\mu \vartheta_{cl}(x) + \frac{\alpha}{\beta^2}\left(\frac{\alpha}{M^2}\right)^{\beta^2/8\pi}(\cos \beta \vartheta_{cl}(x) - 1).$$
(6.11)

This agrees with our expression for the energy density of the ground state of the sine-Gordon model (3.10), where the ultraviolet cut-off is equal to the renormalized mass of the sine-Gordon quanta, $\Lambda = \sqrt{\alpha}$.

The most convenient choice of the renormalization point is $M = \alpha(M)$. This yields

$$\mathcal{L}^{(r)}[\vartheta_{cl}(x)] = \frac{1}{2}\partial_\mu \vartheta_{cl}(x)\partial^\mu \vartheta_s(x) + \frac{\alpha}{\beta^2}(\cos \beta \vartheta_{cl}(x) - 1).$$
(6.12)

Since we have not specified the classical solution, our result is valid for quantum corrections around an arbitrary classical solution of the sine-Gordon model. Our result of the calculation of the quantum fluctuations agrees with that carried out by Korepin *et al* [15].

According to the renormalized Lagrangian (6.10) the soliton (antisoliton) mass is equal to $M_s = M_{\bar{s}} = 8\sqrt{\alpha}/\beta^2$. The masses of breathers would be changed as follows $M_{\text{br}}^{(n)} = (16\sqrt{\alpha}/\beta^2)\sin \nu_n$ with $\nu_n = n\beta^2/16$ and $n = 1, 2, \ldots, 8\pi/\beta^2$.

Hence, quantum fluctuations, calculated with the renormalization prescription expounded above, do not lead to the appearance of a singular point in the sine-Gordon model and allow the continuation of the theory to the region $\beta^2 \geqslant 8\pi$ as has been suspected by Zamolodchikov and Zamolodchikov [20].

As has been pointed out in [1] for $\beta^2 > 8\pi$ the $(1 + 1)$-dimensional world is populated mainly by solitons and antisolitons. Breather states are prohibited for $\beta^2 > 8\pi$. This agrees with the assertion by Zamolodchikov and Zamolodchikov [20], which reads in our notation: 'At $\beta^2 > 8\pi$ all bound states including the "elementary" particle of the sine-Gordon Lagrangian (1.2) become unbound. Thus, at $\beta^2 \geqslant 8\pi$ the spectrum contains solitons and antisolitons only'.

The phase shift for soliton–soliton scattering has been calculated by Weisz in dependence on the rapidity difference θ and the sine-Gordon coupling constant $\lambda > 1$ [22]. For $-\infty < \mathrm{Re}\,\theta < +\infty$ and $|\mathrm{Im}\,\theta| < \min[\pi, \lambda\pi]$ the integral representation for the phase shift reads

$$\delta_{\mathrm{ss}}(\theta) = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{t} \frac{\sin\left(\frac{\theta t}{\pi}\right) \sinh\left(\frac{1}{2}(\lambda - 1)t\right)}{\sin\left(\frac{1}{2}\lambda t\right) \cosh\left(\frac{1}{2}t\right)} \tag{6.13}$$

and 'exhibits the absence of physical bound states for $\lambda > 1$' [22]. In our renormalization procedure expounded above $\lambda = \beta^2/8\pi > 1$.

Thus, the phase shift $\delta_{\mathrm{ss}}(\theta)$, defined by (6.13), should describe soliton–soliton scattering for the sine-Gordon coupling constants obeying the constraint $\beta^2 > 8\pi$.

The absence of contributions from soliton–antisoliton bound states to the phase shift $\delta_{\mathrm{ss}}(\theta)$ of soliton–soliton scattering for $\beta^2 > 8\pi$ agrees with conclusions by Zamolodchikov and Zamolodchikov [20] and ours, concerning the population of the $(1 + 1)$-dimensional world by only solitons and antisolitons for $\beta^2 > 8\pi$.

7. Conclusion

We have shown that the vacuum energy density of the ground state of the sine-Gordon model is bounded from below even for $\beta^2 > 8\pi$. We have found some unconvincing assumptions of Coleman's proof. These are (i) the parameter γ_0, normalizing to zero the classical potential energy of the sine-Gordon model, has been assumed additively renormalizable and set finite after renormalization, (ii) the renormalized Hamiltonian has been found depending on the infrared cut-off μ with divergent contributions in the limit $\mu \to 0$ and (iii) the vacuum energy density (2.21) calculated by Coleman is equal to $\mathcal{E}_{\mathrm{vac}}(M) = -\infty$ in the infrared limit $\mu \to 0$ for any finite scale M and coupling constant β, whereas the sine-Gordon model is well defined in the infrared limit $\mu \to 0$ (see the appendix).

Our direct calculation of the vacuum energy density is non-perturbative and exact. We have shown explicitly that the vacuum energy density of the sine-Gordon model can never be a negative quantity if the potential energy is normalized to zero as is done at the classical level.

Summarizing the obtained results we can conclude that in the region of coupling constants, obeying the constraint $\beta^2 > 8\pi$, the sine-Gordon model can be treated well. For the coupling constants $\beta^2 > 8\pi$ the sine-Gordon model describes only solitons and antisolitons without breathers. The amplitudes of scattering of soliton by soliton and soliton by antisoliton are well defined for $\beta^2 > 8\pi$ without soliton–antisoliton bound state contributions to the intermediate states. In our approach the sine-Gordon model for coupling constants $\beta^2 > 8\pi$ is a quantum field theory with Wightman's observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ [7, 8, 13]. Therefore, it does not contradict Coleman's theorem, asserting the absence of spontaneously broken continuous symmetry in quantum field theories with Wightman's observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$.

Appendix. Non-perturbative renormalizability of the sine-Gordon model

As has been shown in [2] the massless Thirring model with non-vanishing external sources is equivalent to the sine-Gordon model, where the mass of Thirring fermion fields m is considered as an external source $\sigma(x) = -m$ for the scalar fermion density $\bar{\psi}(x)\psi(x)$. Therefore, the properties of non-perturbative renormalizability of the massless Thirring model investigated in [2] should be fully extended to the sine-Gordon (SG) model.

The generating functional of Green functions in the SG model we define as

$$Z_{\text{SG}}[J] = \int \mathcal{D}\vartheta \exp i \int d^2x \left\{ \frac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + \frac{\alpha_0}{\beta^2}(\cos\beta\vartheta(x) - 1) + \vartheta(x)J(x) \right\} \quad \text{(A.1)}$$

where $J(x)$ is an external source of the $\vartheta(x)$-field.

The Lagrangian of the SG model is invariant under the transformations

$$\vartheta(x) \rightarrow \vartheta'(x) = \vartheta(x) + \frac{2\pi n}{\beta} \quad \text{(A.2)}$$

where n is an integer number running over $n = 0, \pm 1, \pm 2, \ldots$. In order to get the generating functional $Z_{\text{SG}}[J]$ invariant under the transformations (A.2) it is sufficient to restrict the class of functions describing the external source of the ϑ-field and impose the constraint [7]

$$\int d^2x \, J(x) = 0. \quad \text{(A.3)}$$

Non-perturbative renormalizability of the SG model we understand as a possibility to remove all divergences by renormalizing the coupling constant α_0. Indeed, since the coupling constant β is related to the coupling constant of the Thirring model g [1, 2] which is unrenormalized $g_0 = g$, so the coupling constant β should possess the same property, i.e. $\beta_0 = \beta$. Hence, only the coupling constant α_0 should undergo renormalization.

The Lagrangian of the SG model written in terms of *bare* quantities reads

$$\mathcal{L}_{\text{SG}}(x) = \frac{1}{2}\partial_\mu \vartheta_0(x)\partial^\mu \vartheta_0(x) + \frac{\alpha_0}{\beta^2}(\cos\beta\vartheta_0(x) - 1). \quad \text{(A.4)}$$

Since β is the unrenormalized coupling constant, the field $\vartheta_0(x)$ should be also unrenormalized, $\vartheta_0(x) = \vartheta(x)$. This means that there is no renormalization of the wavefunction of the ϑ-field. As a result the Lagrangian $\mathcal{L}_{\text{SG}}(x)$ of the SG model in terms of renormalized quantities can be written by

$$\mathcal{L}_{\text{SG}}(x) = \frac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + \frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1) + (Z_1 - 1)\frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1)$$

$$= \frac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + Z_1\frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1) \quad \text{(A.5)}$$

where Z_1 is the renormalization constant of the coupling constant α. The renormalized coupling constant α is related to the *bare* one by the relation

$$\alpha = Z_1^{-1}\alpha_0. \quad \text{(A.6)}$$

Renormalizability of the SG model as well as the Thirring model we understand as the possibility to replace the ultraviolet cut-off Λ by another finite scale M by means of the renormalization constant Z_1 in the limit $\mu \rightarrow 0$. According to the general theory of renormalizations [14] Z_1 should be a function of the coupling constants β, α, the infrared cut-off μ, the ultraviolet cut-off Λ and a finite scale M:

$$Z_1 = Z_1(\beta, \alpha, M; \mu, \Lambda). \quad \text{(A.7)}$$

Now let us proceed to the evaluation of the generating functional (A.1). For this aim expand the integrand of the generating functional $Z_{SG}[J]$ in powers of $\alpha_0 \cos \beta \vartheta(x)$. This gives

$$Z_{SG}[J] = \lim_{\mu \to 0} e^{-i \int d^2 x \frac{\alpha_0}{\beta^2}} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\frac{\alpha_0}{\beta^2}\right)^n \prod_{i=1}^{n} \int d^2 x_i \int \mathcal{D}\vartheta \prod_{i=1}^{n} \cos \beta \vartheta(x_i)$$
$$\times \exp i \int d^2 x \left\{\frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x) - \frac{1}{2} \mu^2 \vartheta^2(x) + \vartheta(x) J(x)\right\}. \tag{A.8}$$

The integration over the ϑ-field can be carried out explicitly and we get

$$Z_{SG}[J] = \lim_{\mu \to 0} e^{-i \int d^2 x \frac{\alpha_0}{\beta^2}} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\frac{\alpha_0}{2\beta^2}\right)^n \sum_{p=0}^{n} \frac{n!}{(n-p)! p!} \prod_{j=1}^{n-p} \prod_{k=1}^{p} \int d^2 x_j \, d^2 y_k$$
$$\times \exp \left\{\frac{1}{2} n \beta^2 i\Delta(0; \mu) + \beta^2 \sum_{j<k}^{n-p} i\Delta(x_j - x_k; \mu) + \beta^2 \sum_{j<k}^{p} i\Delta(y_j - y_k; \mu)\right.$$
$$\left. - \beta^2 \sum_{j=1}^{n-p} \sum_{k=1}^{p} i\Delta(x_j - y_k; \mu)\right\} \exp \left\{\int d^2 x \beta \left[\sum_{j=1}^{n-p} i\Delta(x_j - x; \mu)\right.\right.$$
$$\left.\left. - \sum_{k=1}^{p} i\Delta(y_j - x; \mu)\right] J(x) + \iint d^2 x \, d^2 y \frac{1}{2} J(x) i\Delta(x - y; \mu) J(y)\right\} \tag{A.9}$$

where the causal Green functions $\Delta(x - y; \mu)$ and $\Delta(0; \mu)$ are defined by [2, 7]

$$\Delta(x - y; \mu) = i\theta(x^0 - y^0) D^{(+)}(x - y; \mu) + i\theta(y^0 - x^0) D^{(-)}(y - x; \mu)$$
$$= -\frac{i}{4\pi} \ln[-\mu^2 (x - y)^2 + i0]$$
$$\Delta(0; \mu) = \frac{i}{4\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right).$$

Taking the limit $\mu \to 0$ we reduce the rhs of (A.9) to the form

$$Z_{SG}[J] = e^{-i \int d^2 x \frac{\alpha_0}{\beta^2}} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p!)^2} \left(\frac{\alpha_0}{2\beta^2}\right)^{2p} \prod_{j=1}^{p} \iint d^2 x_j \, d^2 y_j \left[\left(\frac{M^2}{\Lambda^2}\right)^{\beta^2/8\pi}\right]^{2p}$$
$$\times \exp \left\{\frac{\beta^2}{4\pi} \sum_{j<k}^{p} (\ln[-M^2 (x_j - x_k)^2 + i0] + \ln[-M^2 (y_j - y_k)^2 + i0])\right.$$
$$\left. - \frac{\beta^2}{4\pi} \sum_{j=1}^{p} \sum_{k=1}^{p} \ln[-M^2 (x_j - y_k)^2 + i0]\right\}$$
$$\times \exp \left\{\frac{\beta}{4\pi} \int d^2 x \sum_{j=1}^{p} \ln\left[\frac{(x_j - x)^2 + i0}{(y_j - x)^2 + i0}\right] J(x)\right.$$
$$\left. + \frac{1}{8\pi} \iint d^2 x \, d^2 y J(x) \ln[-M^2 (x - y)^2 + i0] J(y)\right\}$$
$$\times \lim_{\mu \to 0} \exp \left\{-\frac{1}{4\pi} \ln\left(\frac{M}{\mu}\right) \left(\int d^2 x \, J(x)\right)^2\right\}. \tag{A.10}$$

Due to the constraint (A.3) the generating functional $Z_{SG}[J]$ does not depend on the infrared cut-off μ. Using (A.3) we get

$$
Z_{SG}[J] = e^{-i\int d^2x \frac{\alpha_0}{\beta^2}} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p!)^2} \left(\frac{\alpha_0}{2\beta^2}\right)^{2p} \prod_{j=1}^{p} \iint d^2x_j\, d^2y_j \left[\left(\frac{M^2}{\Lambda^2}\right)^{\beta^2/8\pi}\right]^{2p}
$$

$$
\times \exp\left\{ \frac{\beta^2}{4\pi} \sum_{j<k}^{p}(\ln[-M^2(x_j-x_k)^2+i0]+\ln[-M^2(y_j-y_k)^2+i0]) \right.
$$

$$
\left. - \frac{\beta^2}{4\pi} \sum_{j=1}^{p}\sum_{k=1}^{p} \ln[-M^2(x_j-y_k)^2+i0]\right\}
$$

$$
\times \exp\left\{ \frac{\beta}{4\pi} \int d^2x \sum_{j=1}^{p} \ln\left[\frac{(x_j-x)^2+i0}{(y_j-x)^2+i0}\right] J(x) \right.
$$

$$
\left. + \frac{1}{8\pi} \iint d^2x\, d^2y\, J(x)\ln[-M^2(x-y)^2+i0]J(y) \right\}. \tag{A.11}
$$

Passing to a renormalized constant α, $\alpha_0 = Z_1\alpha$, we recast the rhs of (A.11) into the form

$$
Z_{SG}[J] = e^{-i\int d^2x Z_1 \frac{\alpha}{\beta^2}} \sum_{p=0}^{\infty} \left[Z_1 \left(\frac{M^2}{\Lambda^2}\right)^{\beta^2/8\pi}\right]^{2p} \frac{(-1)^p}{(p!)^2} \left(\frac{\alpha}{2\beta^2}\right)^{2p} \prod_{j=1}^{p} \iint d^2x_j\, d^2y_j
$$

$$
\times \exp\left\{ \frac{\beta^2}{4\pi} \sum_{j<k}^{p}(\ln[-M^2(x_j-x_k)^2+i0]+\ln[-M^2(y_j-y_k)^2+i0]) \right.
$$

$$
\left. - \frac{\beta^2}{4\pi} \sum_{j=1}^{p}\sum_{k=1}^{p} \ln[-M^2(x_j-y_k)^2+i0]\right\}
$$

$$
\times \exp\left\{ \frac{\beta}{4\pi} \int d^2x \sum_{j=1}^{p} \ln\left[\frac{(x_j-x)^2+i0}{(y_j-x)^2+i0}\right] J(x) \right.
$$

$$
\left. + \frac{1}{8\pi} \iint d^2x\, d^2y\, J(x)\ln[-M^2(x-y)^2+i0]J(y) \right\}. \tag{A.12}
$$

Setting

$$
Z_1 = \left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi} \tag{A.13}
$$

we are left with the dependence of the generating functional $Z_{SG}[J]$ on the ultraviolet cut-off Λ only in the insignificant constant factor

$$
Z_{SG}[J] = e^{-i\int d^2x \frac{\alpha}{\beta^2}\left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi}} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p!)^2} \left(\frac{\alpha}{2\beta^2}\right)^{2p} \prod_{j=1}^{p} \iint d^2x_j\, d^2y_j
$$

$$
\times \exp\left\{ \frac{\beta^2}{4\pi} \sum_{j<k}^{p}(\ln[-M^2(x_j-x_k)^2+i0]+\ln[-M^2(y_j-y_k)^2+i0]) \right.
$$

$$
-\frac{\beta^2}{4\pi} \sum_{j=1}^{p} \sum_{k=1}^{p} \ln[-M^2(x_j - y_k)^2 + \mathrm{i}0] \Bigg\}
$$

$$
\times \exp \Bigg\{ \frac{\beta}{4\pi} \int \mathrm{d}^2 x \sum_{j=1}^{p} \ln \left[\frac{(x_j - x)^2 + \mathrm{i}0}{(y_j - x)^2 + \mathrm{i}0} \right] J(x)
$$

$$
+ \frac{1}{8\pi} \iint \mathrm{d}^2 x \, \mathrm{d}^2 y \, J(x) \ln[-M^2(x - y)^2 + \mathrm{i}0] J(y) \Bigg\} . \tag{A.14}
$$

The generating functional (A.14) is expressed in terms of the renormalized constant α, the constant β and the finite scale M. The ultraviolet cut-off Λ enters only in the insignificant constant factor, which does not affect the result of the evaluation of correlation functions. This factor can be removed by redefinition of the path-integral measure of the generating functional $Z_{\mathrm{SG}}[J]$.

Thus, the generating functional $Z_{\mathrm{SG}}[J]$ (A.14) can be applied to the evaluation of any renormalized correlation function of the SG model. This testifies the complete non-perturbative renormalizability of the SG model.

Using (A.14) we evaluate the partition function $Z_{\mathrm{SG}}[0]$. It is equal to

$$
Z_{\mathrm{SG}}[0] = \mathrm{e}^{-\mathrm{i} \int \mathrm{d}^2 x \, \frac{\alpha}{\beta^2} \left(\frac{\Lambda^2}{M^2} \right)^{\beta^2/8\pi}} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p!)^2} \left(\frac{\alpha}{2\beta^2} \right)^{2p} \prod_{j=1}^{p} \iint \mathrm{d}^2 x_j \, \mathrm{d}^2 y_j
$$

$$
\times \exp \Bigg\{ \frac{\beta^2}{4\pi} \sum_{j<k}^{p} (\ln[-M^2(x_j - x_k)^2 + \mathrm{i}0] + \ln[-M^2(y_j - y_k)^2 + \mathrm{i}0])
$$

$$
- \frac{\beta^2}{4\pi} \sum_{j=1}^{p} \sum_{k=1}^{p} \ln[-M^2(x_j - y_k)^2 + \mathrm{i}0] \Bigg\} . \tag{A.15}
$$

This expression we use for the calculation of the vacuum energy density of the SG model.

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